A Method for the Liapunov Stability Analysis of Force-Free Dynamical Systems

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A new method for the stability analysis of force-free dynamical systems described by simultaneous sets of ordinary and partial differential equations of motion is presented. Such systems, referred to as hybrid, arise naturally in connection with the motion of spinning flexible bodies. The method is based on Liapunov's second method and works directly with the hybrid system of equations. It involves the construction of a Liapunov functional that takes into account automatically motion integrals resulting from the absence of external forces on the system. The general theory is particularly suitable for the stability analysis of torque-free spinning satellites containing distributed elastic members. As an illustration, the case of spin stabilization of a satellite with flexible appendages is solved.

Introduction

THE rotational motion of a torque-free rigid body is known to be stable if the rotation takes place about an axis corresponding to the maximum or minimum moment of inertia, but the motion is unstable if the rotation takes place about the axis of intermediate moment of inertia (see, for example, the text by Meirovitch,[1] Sec. 6.7). The stability of a force-free system of bodies, where the bodies can rotate with different angular velocities, has been investigated by Pringle,[2] who placed special emphasis on dual-spin systems. A matrix formulation of the problems just described has been provided by Likins and Roberson.[3] The effect of elastically connected moving parts on the stability of motion of a rigid body has been studied by Nelson and Meirovitch.[4] The formulations of Refs. 1–4 are in terms of ordinary differential equations and the stability analyses are based on the Liapunov direct method. Flexible parts are represented in the last three references by discrete models.

In one of the first attempts to treat rigorously distributed elastic members, the stability of motion of a spinning symmetric body which is part rigid and part elastic has been investigated by Meirovitch and Nelson.[5] The mathematical formulation in Ref. 5 consists of a set of ordinary differential equations for the rotational motion and another set of partial differential equations describing the elastic displacements. We shall refer to a system of both ordinary and partial differential equations as "hybrid." The hybrid system of Ref. 5 has been reduced to a system consisting entirely of ordinary differential equations by means of the modal analysis, whereby the displacement of a given point in the continuous elastic members is represented by a finite series of appropriate eigenfunctions multiplied by time-dependent generalized coordinates. The stability of the resulting discrete system has been investigated by means of an infinitesimal analysis and the effect of the flexible parts on the motion stability has been displayed in the form of diagrams relating various parameters of the system.

A general and rigorous method for the stability analysis of systems containing distributed elastic parts has been developed by Meirovitch.[6] The method represents an extension of the Liapunov second method and works directly with the hybrid system of differential equations (in the sense defined previously), instead of the common practice of spatial discretization[4,5] or modal truncation.[5] As an application, the case of gravity-gradient stabilization of a satellite with flexible appendages is solved.

The present paper extends the formulation of Ref. 6 to the case in which the system possesses angular momentum integrals. The formulation is somewhat related to the problem of Ref. 3. However, whereas the formulation of Ref. 3 is restricted to discrete systems but can accommodate multispin bodies, the present formulation is confined to single-spin but permits the treatment of distributed elastic systems. This new formulation can be readily used for a large class of problems involving the stability of torque-free flexible satellites. As an illustration, the general theory presented here is applied to the stability analysis of a spinning satellite resembling that of Ref. 5. The power of the method is clearly demonstrated by the fact that it permits the derivation of closed-form stability criteria, in contrast with the criteria of Ref. 5 obtained numerically. In addition, the results derived here are more general in nature, as certain restrictions placed on the system of Ref. 5 have been relaxed. As expected, a comparison of the criteria derived by means of the present method with those derived in Ref. 5 using an infinitesimal analysis reveals that the present method leads to more stringent but more complete stability requirements.

General Problem Formulation

Let us consider a body of total mass m moving relative to an inertial space XYZ, as shown in Fig. 1. The entire body or

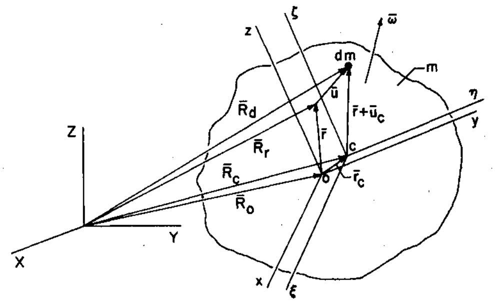

Fig. 1 The flexible body in an inertial space.

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parts of the body are capable of small elastic deformations from a reference equilibrium position coinciding with the undeformed state of the body. Next we define two sets of body axes, the set xyz with the origin at point 0 and coinciding with the principal axes of the body in the undeformed state, and the set $\xi\eta\zeta$ which is parallel to xyz but has the origin at the center of mass c of the deformed body. We note that $\xi\eta\zeta$ is not a principal set of axes. The set xyz serves as a suitable reference frame for measuring elastic deformations whereas the set $\xi\eta\zeta$ is more convenient for expressing the over-all motion. The position of a typical point in the undeformed body relative to axes xyz is denoted by the vector† $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and the elastic displacement of an element of mass dm, originally coincident with that point, by the vector $\mathbf{u} = u(x,y,z,t)\mathbf{i} + v(x,y,z,t)\mathbf{j} + w(x,y,z,t)\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along axes x, y, z (or axes ξ, η, ζ), respectively. The radius vector from point 0 to c is given by $\mathbf{r}_c = m^{-1}\int_m(\mathbf{r} + \mathbf{u})dm = m^{-1}\int_m\mathbf{u}dm$, where we note that $\int_m\mathbf{r}dm$ is zero by virtue of the fact that 0 is the center of mass of the undeformed body. All integrations involved in this paper are carried over the domain occupied by the body in undeformed state, which is designated as the reference state.

From Fig. 1, we conclude that the position of the mass element dm relative to the inertial space is $\mathbf{R}_d = \mathbf{R}_c + \mathbf{r} + \mathbf{u}_c$, where $\mathbf{u}_c = \mathbf{u} - \mathbf{r}_c = u_c\mathbf{i} + v_c\mathbf{j} + w_c\mathbf{k}$ represents the displacement vector measured with respect to axes $\xi\eta\zeta$ and $\mathbf{R}_c$ is the position of the origin of these axes relative to the inertial space. Assuming that axes xyz, hence also axes $\xi\eta\zeta$, rotate with angular velocity $\boldsymbol{\omega} = \omega_\xi\mathbf{i} + \omega_\eta\mathbf{j} + \omega_\zeta\mathbf{k}$ relative to the inertial space, and denoting by $\dot{\mathbf{u}}_c' = \dot{u}_c\mathbf{i} + \dot{v}_c\mathbf{j} + \dot{w}_c\mathbf{k}$ the velocity of dm relative to $\xi\eta\zeta$ due to the elastic effect, it is shown in Ref. 6 that the kinetic energy has the expression

$$T = \tfrac{1}{2}\int_m\dot{\mathbf{R}}_d\cdot\dot{\mathbf{R}}_d dm = \tfrac{1}{2}m\dot{\mathbf{R}}_c\cdot\dot{\mathbf{R}}_c + \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}_d\cdot\boldsymbol{\omega} + [\boldsymbol{\omega}\times\int_m(\mathbf{r}+\mathbf{u}_c)]\cdot\dot{\mathbf{u}}_c'dm + \tfrac{1}{2}\int_m\dot{\mathbf{u}}_c'\cdot\dot{\mathbf{u}}_c'dm \tag{1}$$

where $\mathbf{J}_d$ is the inertia dyadic of the deformed body about axes $\xi\eta\zeta$. The elements of the dyadic are

$$J_{\xi\xi} = \int_m[(y + v_c)^2 + (z + w_c)^2]dm \tag{2a}$$

$$J_{\eta\eta} = \int_m[(x + u_c)^2 + (z + w_c)^2]dm \tag{2b}$$

$$J_{\zeta\zeta} = \int_m[(x + u_c)^2 + (y + v_c)^2]dm \tag{2c}$$

$$J_{\xi\eta} = J_{\eta\xi} = \int_m(x + u_c)(y + v_c)dm \tag{2d}$$

$$J_{\xi\zeta} = J_{\zeta\xi} = \int_m(x + u_c)(z + w_c)dm \tag{2e}$$

$$J_{\eta\zeta} = J_{\zeta\eta} = \int_m(y + v_c)(z + w_c)dm \tag{2f}$$

The kinetic energy can be written conveniently in terms of matrix notation. If $\{\dot{R}_c\}$ is the column matrix corresponding to $\dot{\mathbf{R}}_c$, $\{\omega\}$ the column matrix corresponding to $\boldsymbol{\omega}$, $[J]$ the symmetric matrix whose elements are the elements of the dyadic $\mathbf{J}_d$, and $\{\dot{u}_c'\}$ the matrix representation of the vector $\dot{\mathbf{u}}_c'$, then Eq. (1) can be rewritten in the form

$$T = \tfrac{1}{2}m\{\dot{R}_c\}^T\{\dot{R}_c\} + \tfrac{1}{2}\{\omega\}^T[J]\{\omega\} + \{K\}^T\{\omega\} + \tfrac{1}{2}\int_m\{\dot{u}_c'\}^T\{\dot{u}_c'\}dm \tag{3}$$

where $\{K\}$ is the column matrix with the elements

$$K_\xi = \int_m[(y + v_c)\dot{w}_c - (z + w_c)\dot{v}_c]dm \tag{4a}$$

$$K_\eta = \int_m[(z + w_c)\dot{u}_c - (x + u_c)\dot{w}_c]dm \tag{4b}$$

$$K_\zeta = \int_m[(x + u_c)\dot{v}_c - (y + v_c)\dot{u}_c]dm \tag{4c}$$

The angular velocity components ω_ξ, ω_η, ω_ζ do not represent time rates of change of certain angles but nonintegrable combinations of time derivatives of angular displacements. They are sometimes referred to as time derivatives of quasi-coordinates. Denoting by θ_i and $\dot{\theta}_i$ $(i = 1,2,3)$ the true angular displacements and their time rates of change, the angular velocity vector can be written in the matrix form $\{\omega\} = [\theta]\{\dot{\theta}\}$, where $\{\dot{\theta}\}$ is the column matrix with elements $\dot{\theta}_i (i = 1,2,3)$ and $[\theta]$ is a 3×3 matrix, whose elements depend on the order of the three rotations θ_i used to produce the orientation of axes $\xi\eta\zeta$ relative to an inertial space. In view of this, the kinetic energy can be written in terms of true angular velocities as follows

$$T = \tfrac{1}{2}m\{\dot{R}_c\}^T\{\dot{R}_c\} + \tfrac{1}{2}\{\dot{\theta}\}^T[I]\{\dot{\theta}\} + \{L\}^T\{\dot{\theta}\} + \tfrac{1}{2}\int_m\{\dot{u}_c'\}^T\{\dot{u}_c'\}dm \tag{5}$$

in which the notation

$$[I] = [\theta]^T[J][\theta], \quad \{L\} = [\theta]^T\{K\} \tag{6}$$

has been adopted.

The potential energy arises primarily from two sources, namely gravity and body elasticity. The gravitational potential energy is assumed to be very small compared with the kinetic energy, or the elastic potential energy, and will be ignored. The elastic potential energy, denoted by V_{EL} and referred to at times as strain energy, depends on the nature of the elastic members and is in general a function of the partial derivatives of the elastic displacements u, v, w with respect to the spatial variables x, y, z. Since u_c, v_c, w_c differ from u, v, w by x_c, y_c, z_c, respectively, where the latter are independent of the spatial variables, V_{EL} can be regarded as depending on the partial derivatives of u_c, v_c, w_c with respect to x, y, z. We assume that V_{EL} is a function of $\partial^2u_c/\partial x^2$, $\partial^2u_c/\partial x\partial y$, $\ldots$, $\partial^2w_c/\partial z^2$ but this assumption in no way affects the generality of the formulation. This particular functional dependence of V_{EL} should be regarded as mere scaffolding used in the construction of a general theory, as the final formulation is expressed in a form which involves the partial derivatives only implicitly.

The system differential equations can be obtained by means of Hamilton's principle. To this end, a brief discussion of the generalized coordinates is in order. The motion of the mass center c is generally assumed not to be affected by the motion relative to c, so that it is possible to solve the motion of c independently of the motion relative to c. As a result, the motion of c, referred to as orbital motion, can be regarded as known. We shall confine ourselves to the case in which the first term on the right side of Eq. (5) reduces to a known constant, so that the term can be ignored. This is clearly the case when the orbit is circular, or the motion of c is uniform or zero. It follows that the system generalized coordinates are the three rotations $\theta_i(t)$ and the three elastic displacements $u_c(x,y,z,t)$ $v_c(x,y,z,t)$, $w_c(x,y,z,t)$. The elastic displacements are defined only throughout the domain D_e, namely the subdomain of D corresponding to the elastic continuum, where D is a three-dimensional domain corresponding to the entire body. The domain D_e is bounded by the surface S.

For the holonomic system at hand, Hamilton's principle has the form

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \tag{7}$$

where the motion must be such that the end conditions

$$\delta\theta_1 = \delta\theta_2 = \delta\theta_3 = \delta u_c = \delta v_c = \delta w_c = 0 \quad \text{at} \quad t = t_1, t_2 \tag{8}$$

are satisfied. The integrand L in Eq. (7) is the Lagrangian which has the general functional form

$$L = T - V_{\mathrm{EL}} = \int_D\hat{L}(\theta_i,\dot{\theta}_i,u_c,v_c,\ldots,\dot{w}_c, \partial^2u_c/\partial x^2,\partial^2u_c/\partial x\partial y,\ldots,\partial^2w_c/\partial z^2)dD \tag{9}$$

in which $\hat{L}$ is the Lagrangian density.

An application of Hamilton's principle leads to the system Lagrangian equations of motion. Details of the derivation are given in Ref. 6 and will not be repeated here. Instead we

† Vector quantities are denoted by boldface type.

quote directly from Ref. 6 the ordinary differential equations for the angular displacements

$$\partial L/\partial \theta_i - (d/dt)(\partial L/\partial \dot{\theta}_i) = 0, \quad i = 1,2,3 \tag{10}$$

and the partial differential equations for the elastic displacements

$$\partial \hat{L}/\partial u_c - (\partial/\partial t)(\partial \hat{L}/\partial \dot{u}_c) + \mathcal{L}_{u_c}[u_c,v_c,w_c] + \hat{Q}_{u_c} = 0 \tag{11a}$$

$$\partial \hat{L}/\partial v_c - (\partial/\partial t)(\partial \hat{L}/\partial \dot{v}_c) + \mathcal{L}_{v_c}[u_c,v_c,w_c] + \hat{Q}_{v_c} = 0 \tag{11b}$$

$$\partial \hat{L}/\partial w_c - (\partial/\partial t)(\partial \hat{L}/\partial \dot{w}_c) + \mathcal{L}_{w_c}[u_c,v_c,w_c] + \hat{Q}_{w_c} = 0 \tag{11c}$$

where Eqs. (11) must be satisfied at every point of the domain D_e. Moreover, Eqs. (11) are subject to the boundary conditions

$$\mathbf{B}_j[u_c,v_c,w_c]\cdot\mathbf{B}_k[u_c,v_c,w_c] = 0 \text{ on } S, j = 1,2; \quad k = 3,4 \tag{12}$$

The differential operators $\mathcal{L}(\mathcal{L}_{u_c},\mathcal{L}_{v_c},\mathcal{L}_{w_c})$, $\mathbf{B}_j(B_{ju_c},B_{jv_c},B_{jw_c})$, and $\mathbf{B}_k(B_{ku_c}, B_{kv_c}, B_{kw_c})$ represent vectors defined by the following integration by parts

$$\int_D\left[\frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x^2)}\delta\left(\frac{\partial^2 u_c}{\partial x^2}\right) + \frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x\partial y)}\delta\left(\frac{\partial^2 u_c}{\partial x\partial y}\right) + \ldots + \frac{\partial \hat{L}}{\partial(\partial^2 w_c/\partial z^2)}\delta\left(\frac{\partial^2 w_c}{\partial z^2}\right) dD = \int_{D_e}\mathcal{L}[u_c,v_c,w_c]\cdot\delta\mathbf{u}dD_e + \mathbf{B}_j[u_c,v_c,w_c]\cdot\mathbf{B}_k[u_c,v_c,w_c]\right|_S, j = 1,2; \quad k = 3,4 \tag{13}$$

Since in our case the boundary conditions result from two integrations by parts, only two of the combinations in Eq. (12) must be satisfied at every point of S, where the indices j and k are different for each combination. We note that the partial derivatives $\partial^2u_c/\partial x^2$, $\partial^2u_c/\partial x\partial y,\ldots$, $\partial^2 w_c/\partial z^2$ enter into Eqs. (11) and (12) only implicitly through the differential operator vectors $\mathcal{L}$, $\mathbf{B}_j$, and $\mathbf{B}_k$, thus lending substance to a statement made earlier regarding the generality of the formulation. The quantities $\hat{Q}_{u_c}$, $\hat{Q}_{v_c}$, $\hat{Q}_{w_c}$ represent distributed internal damping forces which depend on the elastic motion alone and not on the rotational motion. It should be pointed out that the damping forces were added afterward, as such forces cannot be treated by means of Hamilton's principle.

Introducing the generalized momenta

$$p_{\theta_i} = \partial L/\partial\dot{\theta}_i, \quad i = 1,2,3$$
$$\hat{p}_{u_c} = \partial \hat{L}/\partial \dot{u}_c, \quad \hat{p}_{v_c} = \partial \hat{L}/\partial \dot{v}_c, \quad \hat{p}_{w_c} = \partial \hat{L}/\partial \dot{w}_c \tag{14}$$

where the latter three are momentum densities, it is shown in Ref. 6 that the second-order Lagrangian equations, Eqs. (10) and (11), can be converted into twice the number of first-order Hamiltonian equations having the form

$$\dot{\theta}_i = \partial H/\partial p_{\theta_i}, \quad \dot{p}_{\theta_i} = -\partial H/\partial\theta_i, \quad i = 1,2,3 \tag{15a}$$

$$\begin{aligned} \dot{u}_c &= \partial\hat{H}/\partial\hat{p}_{u_c}, \quad \dot{v}_c = \partial\hat{H}/\partial p_{v_c}, \quad \dot{w}_c = \partial\hat{H}/\partial\hat{p}_{w_c} \\ \dot{\hat{p}}_{u_c} &= -\partial\hat{H}/\partial u_c + \mathcal{L}_{u_c}(u_c,v_c,w_c) + \hat{Q}_{u_c} \\ \dot{\hat{p}}_{v_c} &= -\partial\hat{H}/\partial v_c + \mathcal{L}_{v_c}(u_c,v_c,w_c) + \hat{Q}_{v_c} \\ \dot{\hat{p}}_{w_c} &= -\partial\hat{H}/\partial w_c + \mathcal{L}_{w_c}(u_c,v_c,w_c) + \hat{Q}_{w_c} \end{aligned} \tag{15b}$$

where Eqs. (15b) must be satisfied at every point of D_e. In Eqs. (15), H is the Hamiltonian defined by

$$H = \sum_{i=1}^{3} p_{\theta_i}\dot{\theta}_i + \int_{D_e}(\hat{p}_{u_c}\dot{u}_c + \hat{p}_{v_c}\dot{v}_c + \hat{p}_{w_c}\dot{w}_c)dD_e - L \tag{16}$$

and $\hat{H}$ is the corresponding Hamiltonian density. It should be noticed here that the Hamiltonian has a hybrid form as it is a function and a functional at the same time. The equations for the elastic motion are subject to the same boundary conditions, Eqs. (12). When the kinetic energy is quadratic in the generalized velocities, the Hamiltonian reduces to the form

$$H = T + V_{\mathrm{EL}} \tag{17}$$

which is recognized as the system total energy.

Stability Analysis

A general and rigorous method for the stability analysis of hybrid systems of equations has been developed in Ref. 6. We shall not present all the details here but only summarize the main features.

Consider the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ in which $\mathbf{x}$ represents an element in a space which for a hybrid system can be regarded as the cartesian product of a finite dimensional vector space and a function space, and define a scalar function $U(\mathbf{x})$ such that $U(\mathbf{0}) = 0$. The total time derivative of U along a trajectory of the system is defined by $\dot{U} = dU/dt = \nabla U\cdot\dot{\mathbf{x}} = \nabla U\cdot\mathbf{X}$. Next consider the following theorems:

Theorem I: If there exists for the system a positive (negative) definite function $U(\mathbf{x})$ whose total time derivative $\dot{U}(\mathbf{x})$ is negative (positive) semidefinite along every trajectory of the system, then the trivial solution $\mathbf{x} = \mathbf{0}$ is *stable.*

Theorem II: If the conditions of Theorem I are satisfied and if, in addition, the set of points at which $\dot{U}(\mathbf{x})$ is zero contains no nontrivial positive half-trajectory, then the trivial solution is *asymptotically stable.*

Theorem III: If there exists for the system a function $U(\mathbf{x})$ whose total time derivative $\dot{U}(\mathbf{x})$ is positive (negative) definite along every trajectory of the system and the function itself can assume positive (negative) values in the neighborhood of the origin, then the trivial solution is *unstable.*

Theorem IV: Suppose that a function $U(\mathbf{x})$ such as in Theorem III exists but for which $\dot{U}(\mathbf{x})$ is only positive (negative) semidefinite and, in addition, the set of points at which $\dot{U}(\mathbf{x})$ is zero contains no nontrivial positive half-trajectory. Moreover, in every neighborhood of the origin there is a point $\mathbf{x}_0$ such that for arbitrary $t_0 > 0$ we have $U(\mathbf{x}_0) > 0(<0)$. Then the trivial solution is *unstable* and the trajectories $\mathbf{x}(\mathbf{x}_0, t_0, t)$ for which $U(\mathbf{x}_0) > 0(<0)$ must leave the open domain $\|\mathbf{x}\| < \epsilon$ as the time t increases.

A function U satisfying any of the preceding theorems is referred to as a Liapunov function. For a hybrid system it is both a function and a functional simultaneously but will be referred to as a Liapunov functional.

Using the results derived in the section General Problem Formulation, it is not difficult to show that

$$\dot{H} = \int_{D_e}(\hat{Q}_{u_c}\dot{u}_c + \hat{Q}_{v_c}\dot{v}_c + \hat{Q}_{w_c}\dot{w}_c)dD_e \tag{18}$$

It is reasonable to assume that the damping forces $\hat{Q}_{u_c}$, $\hat{Q}_{v_c}$, $\hat{Q}_{w_c}$ are such that $\dot{H}$ is negative semidefinite

$$\dot{H} \leq 0 \tag{19}$$

Moreover, due to coupling, the damping forces are never identically zero at every point of the phase space except at an equilibrium point, where the phase space, denoted symbolically by S, is the space defined by the generalized coordinates and the conjugate momenta, or alternatively the generalized coordinates and velocities. In view of this, the Hamiltonian may qualify as a Liapunov functional. Indeed, according to Theorem II, if H is positive definite at an equilibrium point, then the equilibrium is *asymptotically stable* and H can be regarded as a Liapunov functional. On the other hand, if H is not positive definite and there are points for which it is negative, then by Theorem IV the equilibrium point is *unstable.*

The equilibrium points of the system are the solution of the equations

$$\partial H/\partial p_{\theta_i} = 0, \quad -\partial H/\partial\theta_i = 0, \quad i = 1,2,3 \tag{20a}$$

$$\begin{aligned} \partial\hat{H}/\partial\hat{p}_{u_c} = \partial\hat{H}/\partial\hat{p}_{v_c} &= \partial\hat{H}/\partial\hat{p}_{w_c} = 0 \\ -\partial\hat{H}/\partial u_c + \mathcal{L}_{u_c}[u_c,v_c,w_c] &= 0 \\ -\partial\hat{H}/\partial v_c + \mathcal{L}_{vc}[u_c,v_c,w_c] &= 0 \\ -\partial\hat{H}/\partial w_c + \mathcal{L}_{w_c}[u_c,v_c,w_c] &= 0 \end{aligned} \tag{20b}$$

where Eqs. (20b) must be satisfied at every point of D_e.

Assuming that in the elastic potential energy the displacements u, v, w (hence u_c, v_c, w_c) are independent of one another, it is shown in Ref. 6 that

$$V_{EL} = \tfrac{1}{2}\int_{D_e} u\mathcal{L}_u[u] + v\mathcal{L}_v[v] + w\mathcal{L}_w[w]dD_e \geq \tfrac{1}{2}\int_{D_e}\rho(\Lambda_{1u}{}^2u_c{}^2 + \Lambda_{1v}{}^2v_c{}^2 + \Lambda_{1w}{}^2w_c{}^2)dD_e \quad (21)$$

where ρ is the mass density and $\Lambda_{1u}{}^2, \Lambda_{1v}{}^2, \Lambda_{1w}{}^2$ are the lowest eigenvalues associated with the vibrations u, v, w, respectively.

Hence, let us introduce the functional

$$\kappa = T + \tfrac{1}{2}\int_{D_e}\rho(\Lambda_{1u}{}^2u_c{}^2 + \Lambda_{1v}{}^2v_c{}^2 + \Lambda_{1w}{}^2w_c{}^2)dD_e \quad (22)$$

Now, because $H \geq \kappa$, it is sufficient to show that κ is positive definite for the system to be asymptotically stable.

Systems Free of External Forces

When there are no motion integrals, the state at time t of the hybrid system considered is given by an element in a space S which can be regarded as the cartesian product of the finite dimensional vector space defined by θ_i, $p_{\theta i}$ $(i = 1,2,3)$ and the function space defined by $u_c, v_c, w_c, \hat{p}_{uc}, \hat{p}_{vc}, \hat{p}_{wc}$. The space S is simply the phase space. Alternatively, the space can be regarded as the cartesian product of the vector space defined by θ_i, $\dot{\theta}_i (i = 1,2,3)$ and the function space defined by $u_c, v_c, w_c, \dot{u}_c, \dot{v}_c, \dot{w}_c$. The motion of the system can be interpreted as a continuous mapping of the space S onto itself. This implies that if the state of the system at a given time is known, then the state is known for any subsequent time.

Under certain circumstances the system possesses motion integrals. For example, such integrals occur when the system is free of external forces, in which case the motion integrals are simply momentum integrals. These integrals can be regarded as constraint equations relating the system velocities. Constraints may be interpreted as restricting the motion to a subspace of a correspondingly smaller dimension.

Let us assume that the system considered is free of external forces, so that the three torque components about the mass center c are zero. It follows that the angular momentum vector about c is conserved

$$\mathbf{L}_c = \int_m(\mathbf{r} + \mathbf{u}_c) \times [\dot{\mathbf{u}}_c' + \boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}_c)]dm = \boldsymbol{\beta} = \text{const} \quad (23)$$

in which $\boldsymbol{\beta}$ denotes the constant angular momentum vector. In matrix notation, Eq. (23) assumes the form

$$[J]\{\omega\} + \{K\} = \{\beta\} \quad (24)$$

where $[J]$ is the inertia matrix of the deformed body, namely, the matrix representation of the inertia dyadic whose elements are given by Eqs. (2), and $\{K\}$ is the column matrix of the angular momentum components due to the elastic motion; the elements of $\{K\}$ are given by Eqs. (4). Clearly, $\{\beta\}$ is the matrix representation of the vector $\boldsymbol{\beta}$.

Equation (24) can be used to eliminate the angular velocities $\dot{\theta}_i (i = 1,2,3)$ from the kinetic energy. Indeed, premultiplying Eq. (24) by $[J]^{-1}$ and rearranging, we obtain

$$\{\omega\} = [J]^{-1}\{\beta - K\} \quad (25)$$

Introducing Eq. (25) into Eq. (3), and ignoring the term due to the orbital motion, we can write the kinetic energy in the form

$$T = T_2 + T_0 \quad (26)$$

in which

$$T_2 = \tfrac{1}{2}\int_m\{\dot{u}_c'\}^T\{\dot{u}_c'\}dm - \tfrac{1}{2}\{K\}^T[J]^{-1}\{K\} \quad (27)$$

is a quadratic expression in the elastic velocities $\dot{u}_c, \dot{v}_c, \dot{w}_c$, and

$$T_0 = \tfrac{1}{2}\{\beta\}^T[J]^{-1}\{\beta\} \quad (28)$$

is an expression in the angular coordinates and elastic displacements alone, hence it contains no velocities. It turns out that not all three angular coordinates are present in T_0 but only two of them. To show this, we denote by β_0 the magnitude of the initial angular momentum vector, assume for convenience that the direction of the angular momentum vector coincides initially with the inertial axis Z, and express the angular momentum matrix $\{\beta\}$ in the form $\beta_0\{l\}$, where $\{l\}$ is the column matrix of the direction cosines $l_{\xi Z}$, $l_{\eta Z}$, $l_{\zeta Z}$ between Z and axes ξ, η, ζ, respectively. These direction cosines can be expressed in terms of only two angular coordinates.

Inserting Eq. (26), in conjunction with expressions (27) and (28), into Eq. (22), we conclude that the functional κ can be written in the form

$$\kappa = \kappa_1 + \kappa_2 \quad (29)$$

in which $\kappa_1 = T_2$ and

$$\kappa_2 = T_0 + \tfrac{1}{2}\int_{D_e}\rho(\Lambda_{1u}{}^2u_c{}^2 + \Lambda_{1v}{}^2v_c{}^2 + \Lambda_{1w}{}^2w_c{}^2)dD_e = \tfrac{1}{2}\beta_0{}^2\{l\}^T[J]^{-1}\{l\} + \tfrac{1}{2}\int_{D_e}\rho\{u_c\}^T[\Lambda_1{}^2]\{u_c\}dD_e \quad (30)$$

where $\{u_c\}$ is the column matrix of the elastic displacements u_c, v_c, w_c and $[\Lambda_1{}^2]$ is a diagonal matrix of the lowest eigenvalues associated with these displacements. The functional $\hat{\kappa}_2$ can be regarded as a *modified dynamic potential*. By virtue of inequality (21), we conclude that κ_2 is in general smaller than (or equal to) the ordinary dynamic potential $T_0 + V_{EL}$.

But for T to be positive definite for all $\{\beta\}$, T_2 must be positive definite for all $\{u_c\}$. Hence, we conclude that to show that κ is positive definite it is sufficient to show that κ_2 is positive definite. To establish the positive definiteness of κ_2, we consider the possibility of constructing a density function κ_2 associated with κ_2 for every point of the elastic domain D_e and attempt to prove that the function is positive definite at every such point. The conditions for the positive definiteness thus obtained are generally more stringent than necessary.

To obtain the testing density function $\hat{\kappa}_2$, we recall that the inertia matrix $[J]$ of the deformed body contains the elastic displacements u_c, v_c w_c, which are assumed to be small. Denoting by $[J]_0$ the inertia matrix of the body in undeformed state and by $[J]_1$ the change in the inertia matrix because of the elastic deformations, we have

$$[J] = [J]_0 + [J]_1 \quad (31)$$

Since $[J]_1$ is small compared with $[J]_0$, it is not difficult to show that

$$[K] = [J]^{-1} \cong [J]_0{}^{-1} - [J]_0{}^{-1}[J]_1[J]_0{}^{-1} + [J]_0{}^{-1}[J]_1[J]_0{}^{-1}[J]_1[J]_0{}^{-1} \quad (32)$$

where $[K]$ denotes the reciprocal of $[J]$. We shall be concerned with the case in which it is possible to define the density matrix $[\hat{K}]$ satisfying the relation

$$[K] = \int_{D_e}[\hat{K}]dD_e \quad (33)$$

so that the testing density function can be written in the form

$$\hat{\kappa}_2 = \tfrac{1}{2}\beta_0{}^2\{l\}^T[\hat{K}]\{l\} + \tfrac{1}{2}\rho\{u_c\}^T[\Lambda_1{}^2]\{u_c\} \quad (34)$$

To test the density function $\hat{\kappa}_2$ for positive definiteness in the neighborhood of an equilibrium point E, we define the 5×5 Hessian density matrix $[\hat{\mathcal{H}}]_E$ as the matrix of the coefficients associated with the quadratic form $\hat{\kappa}_2|_E$, where the latter is simply the value of $\hat{\kappa}_2$ in the neighborhood of the equilibrium point in question. The positive definiteness of $[\hat{\mathcal{H}}]_E$ can be ascertained by means of Sylvester's criterion (see Ref. 1, Sec. 6.7).

At this point it is appropriate to mention that it is not always possible to construct the density function $\hat{\kappa}_2$. In some cases the difficulty lies in defining a density matrix $[\hat{K}]$, difficulty which can be traced to the last term in Eq. (32). However, this problem may be overcome at times by judicious use of Schwarz's inequality for functions. In other cases the difficulty is caused by coupling of the elastic displacements. Such coupling can be introduced by the motion of the mass center of the deformed body relative to the undeformed state. It must be pointed out, however, that in many analyses the motion of the mass center is ignored by virtue of the assumption of antisymmetric motion. Hence, when the assumption of antisymmetric motion can be safely made, this difficulty can be circumvented with ease. The question remains as to the procedure to be used when problems encountered in the construction of the density function k_2 cannot be overcome. In such cases it may be advisable to use modal analysis in conjunction with series truncation (see, for example, Ref. 5) in order to derive a Hessian matrix $[\mathcal{H}]_E$. To this end, we refer to Eqs. (17) and (26), consider the testing function $H_2 = T_0 + V_{EL}$, solve the eigenvalue problem associated with the vibrations u, v, w, and represent these displacements by finite series of corresponding eigenfunctions multiplying generalized coordinates. However, whereas the Hessian matrix derived from Eq. (34) is a 5×5 density matrix, the Hessian matrix obtained by modal analysis is not a density matrix and its order is at least 5×5, the order depending on the number of eigenfunctions used in the series representing the elastic displacements. It must be stressed that in general it is much more laborious to obtain the Hessian matrix by modal analysis than the Hessian density matrix associated with Eq. (34). This statement is particularly true when the eigenvalue problem associated with u, v, w cannot be readily solved.

Stability of High-Spin Motion of a Flexible Satellite

As an illustration of the method presented, we shall investigate the stability of a spinning satellite simulated by a rigid body with two flexible thin rods, as shown in Fig. 2a. In undeformed state the body possesses principal moments of inertia A, B, C about axes x, y, z, respectively, and the rods are aligned with the z axis. The body is initially spinning undeformed about axis z with angular velocity Ω_s. The problem resembles that of Ref. 5 but, by contrast, the solution of Ref. 5 is obtained by means of an infinitesimal analysis under restrictive circumstances. Specifically, the body considered in Ref. 5 possesses equal moments of inertia about axes x and y and, moreover, the elastic motion is assumed to be antisymmetric, so that point c coincides with point 0 at all times. No such restrictions are placed here. The mathematical model is the same as that used in the illustrative example of Ref. 6. However, whereas in Ref. 6 the case of gravity-gradient stabilization was investigated, here the spin stabilization is being considered.

The domain of the elastic continuum is simply $D_e: h < z < h + l, -(h + l) < z < -h$. Hence, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over $D - D_e$ and $\mathbf{r} = z\mathbf{k}$ over D_e. Assuming only flexural transverse vibration, it follows that $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$, so that $\mathbf{u}_c = u_c\mathbf{i} + v_c\mathbf{j}$ and $\mathbf{r}_c = x_c\mathbf{i} + y_c\mathbf{j}$. From Eqs. (2) we conclude that

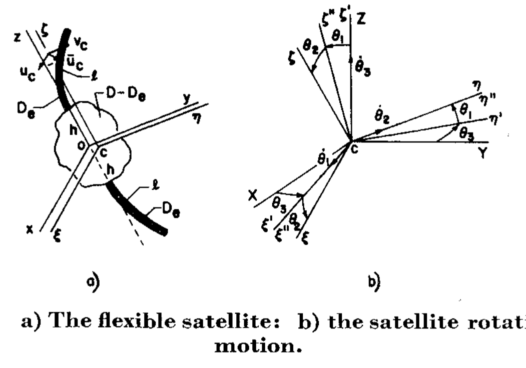

Fig. 2 a) The flexible satellite: b) the satellite rotational motion.

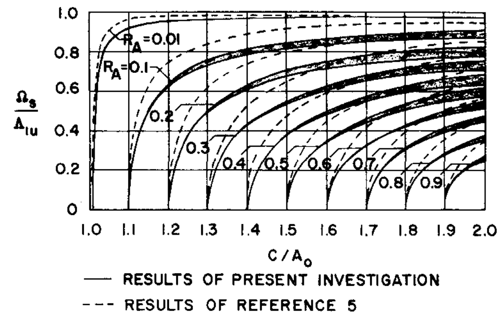

Fig. 3 Stability regions in the parameter plane.

the moments and products of inertia of the deformed body have the values

$$
\begin{aligned}
J_{\xi\xi} &= A + \int_{D_e}\rho v_c^2 dz, \quad J_{\xi\eta} = \int_{D_e}\rho u_c v_c dz \\
J_{\eta\eta} &= B + \int_{D_e}\rho u_c^2 dz, \quad J_{\xi\zeta} = \int_{D_e}\rho z u_c dz \\
J_{\zeta\zeta} &= C + \int_{D_e}\rho (u_c^2 + v_c^2) dz, \quad J_{\eta\zeta} = \int_{D_e}\rho z v_c dz
\end{aligned}
\tag{35}
$$

so that

$$
[J]_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}
$$
$$
[J]_1 = \begin{bmatrix} \int\rho v_c^2 dz & -\int\rho u_c v_c dz & -\int\rho z u_c dz \\ -\int\rho u_c v_c dz & \int\rho u_c^2 dz & -\int\rho z v_c dz \\ -\int\rho z u_c dz & -\int\rho z v_c dz & \int\rho(u_c^2 + v_c^2) dz \end{bmatrix}
\tag{36}
$$

where all integrations in $[J]_1$ are carried over D_e.

We are interested in the stability of pure spin about axis z. The equilibrium configuration corresponding to pure spin is given by

$$E: \theta_i = 0 \ (i = 1,2) \quad \text{and} \quad u_c = v_c = 0$$

Since in the equilibrium configuration the body spins about axis z with angular velocity Ω_s, where z coincides with the inertial axis Z, it follows that $\beta_0 = C\Omega_s$. Moreover, from Fig. 2b we conclude that the direction cosines have the values $l_{\xi Z} = -\cos\theta_1 \sin\theta_2$, $l_{\eta Z} = \sin\theta_1$, and $l_{\zeta Z} = \cos\theta_1 \cos\theta_2$. Introducing all these values into the first term in Eq. (30), considering Eq. (32), and ignoring terms in θ_1, θ_2, u_c, and v_c of order larger than two (as well as constant terms), we can write

$$
\begin{aligned}
\beta_0^2\{l\}^T[K]\{l\}|_E = \Omega_s^2[&(C/B)(C - B)\theta_1^2 + \\
&(C/A)(C - A)\theta_2^2 - \int_{D_e}\rho(u_c^2 + v_c^2)dz + \\
&(1/A)(\int_{D_e}\rho z u_c dz)^2 + (1/B)(\int_{D_e}\rho z v_c dz)^2 - \\
&2(C/A)\theta_2\int_{D_e}\rho z u_c dz + 2(C/B)\theta_1\int_{D_e} z v_c dz]
\end{aligned}
$$

We notice that in the above expression there are two terms involving squares of the integrals $\int_{D_e}\rho z u_c dz$ and $\int_{D_e}\rho z v_c dz$, which prevents a ready definition of a density function. Using Schwarz's inequality for functions, we have

$$0 \le (\int_{D_e}\rho z u_c dz)^2 \le \int_{D_e}\rho z^2 dz \int_{D_e}\rho u_c^2 dz$$

where the first equality sign holds if u_c is symmetric in z and the second one when u_c is proportional to z. Since in rotational motion the elastic displacement u_c is not likely to remain symmetric at all times, and, moreover, u_c cannot be proportional to z, it follows that both equality signs must be ignored. Introducing the notation $A_1 = \int_{D_e}\rho z^2 dz$, where A_1 is recognized as the moment of inertia about axis x of the mass associated with the elastic continuum when in undeformed state, we can write

$$(\int_{D_e}\rho z u_c dz)^2 = \alpha_u A_1 \int_{D_e}\rho u_c^2 dz, \quad 0 < \alpha_u < 1$$

Similarly, we have

$$(\int_{D_e}\rho z v_c dz)^2 = \alpha_v B_1 \int_{D_e}\rho v_c^2 dz, \quad 0 < \alpha_v < 1$$

Although $B_1 = A_1$, we introduced a different symbol to keep the notation consistent. Inserting these expressions into Eq. (34) evaluated at E, we arrive at the density function

$$2\hat{k}_2|_E = \Omega_s^2[(C/B)(\hat{C} - \hat{B})\theta_1^2 + (C/A)(\hat{C} - \hat{A})\theta_2^2 + 2(C/B)\rho z\theta_1 v_c - 2(C/A)\rho z\theta_2 u_c - \rho(1 - \alpha_u A_1/A)u_c^2 - \rho(1 - \alpha_v B_1/B)v_c^2] + \rho(\Lambda_{1u}^2 u_c^2 + \Lambda_{1v}^2 v_c^2) \quad (37)$$

where $\hat{A} = A/D_e$, $\hat{B} = B/D_e$, and $\hat{C} = C/D_e$, in which $D_e = 2l$. Expression (37) leads to the Hessian density matrix

$$[\hat{\mathcal{K}}]_E = \begin{bmatrix} \frac{C}{B}(\hat{C} - \hat{B})\Omega_s^2 & 0 & 0 & \frac{C}{B}\rho z\Omega_s^2 \\ 0 & \frac{C}{A}(\hat{C} - \hat{A})\Omega_s^2 & -\frac{C}{A}\rho z\Omega_s^2 & 0 \\ 0 & -\frac{C}{A}\rho z\Omega_s^2 & \rho\left[\Lambda_{1u}^2 - \Omega_s^2\left(1 - \alpha_u\frac{A_1}{A}\right)\right] & 0 \\ \frac{C}{B}\rho z\Omega_s^2 & 0 & 0 & \rho\left[\Lambda_{1v}^2 - \Omega_s^2\left(1 - \alpha_v\frac{B_1}{B}\right)\right] \end{bmatrix} \quad (38)$$

Using Sylvester's criterion, we conclude that the matrix $[\hat{\mathcal{K}}]_E$ is positive definite if the following inequalities are satisfied

$$C > B, \quad C > A \quad (39a)$$

$$\Lambda_{1u}^2 > \Omega_s^2[1 - \alpha_u A_1/A + (C/A)\, 2l(\rho z^2)_{\max}/(C - A)]$$
$$\Lambda_{1v}^2 > \Omega_s^2[1 - \alpha_v B_1/B + (C/B) 2l(\rho z^2)_{\max}/(C - B)] \quad (39b)$$

Two major conclusions can be drawn from inequalities (39):

1) For spin stabilization the spinning motion should be imparted about the axis of maximum moment of inertia.

2) Spin stabilization is possible if the initial unperturbed spin Ω_s is smaller than the first natural frequency associated with the vibration of the rods in both x and y directions. For a given body, the value of Ω_s required for stability is dictated by the mass distribution of the elastic members according to inequalities (39b). The satisfaction of inequalities (39b) is virtually ensured if the rods are very stiff.

These conclusions are appreciably more general than those reached in Ref. 5 because the present analysis is not restricted to bodies for which $B = A$ or to antisymmetric elastic deformations, as the analysis of Ref. 5 is. Moreover, since criteria (39) are in closed form they lend themselves to an easy physical interpretation, which furnishes a more complete picture of the stability problem and leaves a numerical solution at a clear disadvantage. Criteria (39) are in general more stringent than necesssary since the present method leads to sufficient but not necessary conditions for stability. This is so because testing the positive definiteness of H is replaced by the testing of the functional κ, Eq. (22), where $H \geq \kappa$. Moreover, the testing of κ for positive definiteness is further replaced by the testing of the density $\hat{k}_2$. If we were to plot the regions of stability in a parameter space, then the actual regions of stability would always be greater than (or equal to) the stability regions obtained by using conditions (39). Hence, if inequalities (39) are satisfied we can be sure of the motion stability.

To develop a better appreciation of the problem, we consider the case of uniform rods. The assumption of uniform rods does not affect criteria (39a). On the other hand, denoting by A_0 and B_0 the moments of inertia of the rigid body alone, namely, the body occupying the domain $D - D_e$, and by $m_e = \rho l$ the mass of one rod, inequalities (39b) can be rewritten in the form

$$\frac{\Omega_s}{\Lambda_{1u}} < \left[1 - \frac{\alpha_u R_A}{1 + R_A} + \frac{C}{A_0(1 + R_A)}\frac{2m_e(h + l)^2}{C - A_0(1 + R_A)}\right]^{-1/2}$$
$$\frac{\Omega_s}{\Lambda_{1v}} < \left[1 - \frac{\alpha_v R_B}{1 + R_B} + \frac{C}{B_0(1 + R_B)}\frac{2m_e(h + l)^2}{C - B_0(1 + R_B)}\right]^{-1/2} \quad (40)$$

where $R_A = A_1/A_0$ and $R_B = B_1/B_0$. To interpret the first of inequalities (40), we plot the curves Ω_s/Λ_{1u} vs C/A_0 obtained by replacing the inequality sign by the equal sign and regarding R_A as a parameter. Figure 3 shows plots for $\alpha_u = 0$, 1 and $h = 2l$. For a given value of R_A, the curve corresponding to a certain value of α_u provides the boundary of the stability region, where the stability region is that below the curve. The stability region increases with increasing α_u. For a given R_A, the boundary curve lies somewhere between the curves corresponding to $\alpha_u = 0$ and $\alpha_u = 1$, namely, in the shaded region bounded by these curves. On physical grounds, the actual boundary curve cannot coincide with either of the curves $\alpha_u = 0$ or $\alpha_u = 1$. However, because the shaded region is sufficiently narrow compared with the corresponding stability region, the actual value of α_u does not appear to be particularly significant. The second of inequalities (40) yields a diagram identical in every respect to Fig. 3 but with Λ_{1v}, α_v, B_0, and R_B replacing Λ_{1u}, α_u, A_0, and R_A, respectively.

Comparing Fig. 3 with Fig. 2 of Ref. 5 (note that the parameters were chosen so that the two diagrams are comparable), it is clear that there is general agreement between the closed-form criteria obtained here and the numerical results of Ref. 5. However, the regions of stability obtained here are smaller than those of Ref. 5, which is to be expected because the present method leads in general to more stringent stability requirements than an infinitesimal analysis would.

Summary and Conclusions

A new method for the stability analysis of hybrid systems possessing motion integrals is presented. The method is based on the Liapunov direct method and considers for testing purposes a hybrid expression, namely an expression which is both a function and a functional simultaneously. The procedure is suitable for the stability investigation of bodies which are part rigid and part elastic and or which there exists momentum integrals. It involves the construction of a Liapunov functional which takes into account automatically the momentum integrals, thus reducing the number of generalized coordinates. The method represents an extension of the theory developed by the author in Ref. 6 and it is as general as it is powerful. When the application of the method is possible, closed-form stability criteria can be derived with a minimum amount of effort. The theory is suitable for the stability analysis of spinning flexible bodies free of external torques.

As an application of the general theory, the case of torque-free spinning satellite with flexible antennas, simulated by thin rods, is solved. The analysis yields a set of stability criteria involving the system parameters such as the body moments of inertia, the length and mass distribution of the elastic rods, the lowest natural frequencies of the rods, and the satellite spin velocity. The power of the method is illustrated by the relative ease with which closed-form stability criteria are derived and by the amount of information which can be extracted from their ready physical interpretation. In particular, the analysis shows that, for stability, the spinning motion is to be imparted about the axis of maximum moment of inertia. This is the well-known "greatest moment of inertia" requirement. Moreover, the initial spin velocity Ω_c should not be merely lower than the first natural frequencies Λ_{1u} and Λ_{1v} associated with the transverse vibration of the rods (as the frequency of simple harmonic excitation of the rods should be if resonance is to be prevented), but the ratios Ω_s/Λ_{1u} and Ω_s/Λ_{1v} are dictated by the system parameters. Of course, for very stiff rods the natural frequencies Λ_{1u} and Λ_{1v} may be sufficiently high that the satisfaction of criteria (40) is ensured.

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Buckling of a Thin Annular Plate under Uniform Compression

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The buckling of a circular annular plate with the outer edge clamped, the inner edge free, and loaded with uniform radial compressive force applied at the outside edge has been studied both theoretically and experimentally. Solution to the differential equation for buckling has been sought in the form $w = A_n(r) \cos n\theta$, $n = 0,1,2,\ldots$. The differential equation has been solved exactly for $n = 0$ and $n = 1$ and approximately for higher values of n as well as for $n = 0$ and $n = 1$. The solutions indicate that for small ratios of inner to outer radius the plate buckles into a radially symmetric mode. When the ratio of the inner to outer radius exceeds a certain value, the minimum buckling load corresponds to buckling modes with waves along the circumference. The number of waves depends on the ratio of the inner and the outer radii. Tests were carried out with thin aluminum plates, and the results corroborate the theoretical predictions.

Nomenclature

a = outer radius
b = inner radius
D = bending stiffness of the plate = $Eh^3/[12(1 - \nu^2)]$
E = modulus of elasticity of the plate
h = thickness of plate
T = temperature rise above ambient
V = potential energy
w = transverse displacement perturbation
ν = Poisson's ratio
α_A, α_S = coefficient of thermal expansion of aluminum and steel, respectively
θ_c = theoretical rise of temperature above ambient for buckling
N_0 = radial compressive force at the outer edge
N_{0cr} = radial compressive force at the outer edge at buckling
N_r = radial stress resultant
N_θ = circumferential stress resultant
$N_{r\theta}$ = shear stress resultant
T_c = experimentally observed rise of temperature above ambient for buckling

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Introduction

THE elastic stability of a thin circular plate was studied first by Bryan[1] in 1891. He showed that the minimum buckling load for a circular plate without a central hole corresponds to a radially symmetric buckling mode. The buckling of a circular annular plate subjected to shearing forces distributed along the edges was first studied by Dean.[2] Since then, many researchers have investigated the buckling of a circular annular plate subjected to various loading conditions. Willers[3] considered the case of a plate subjected to bending moment caused by initial stresses. Some of these cases have been extended to plates with varying thickness.[4–7] The buckling of a thin circular annular plate subjected to equal compressive loadings at both the edges has been studied by Olsson,[8] Schubert,[9] and Yamaki.[10] Olsson and Schubert considered only radially symmetric buckling modes. Yamaki showed that, for some cases, a radially symmetric buckling mode does not correspond to the lowest buckling load.

The buckling of a circular annular plate clamped at the outer edge with the inner edge free and subjected to uniform radial compression at the outer edge was first studied by Meissner.[11] He assumed a radially symmetric buckling mode and obtained a relationship between the buckling load and the ratio between the inner and the outer radii.